Short Simple Geodesic Loops on a 2-Sphere

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Background: Definitions

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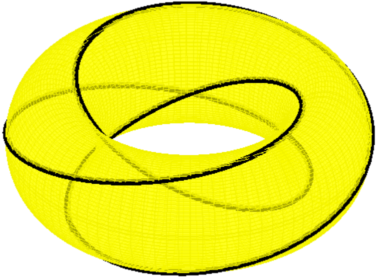
Examples:

- straight line segments in Euclidean space
- great circle arcs on the sphere
- this periodic curve on the torus



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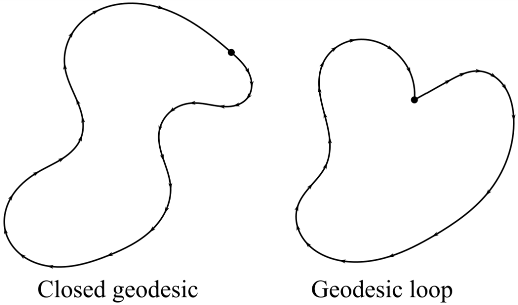

Geodesic Loops

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Answer

On a closed surface of diameter d, between any points p, q there are k geodesics of length at most...

- $22kd$, or $20kd$ if $p = q$ (A. Nabutovsky and R. Rotman 2002, 2011 [4]).
- $8kd$, or $6kd$ if $p = q$ (H. Y. Cheng 2022 [1]),

In dimension n, the known bound is $\leq 4nk^2d$ (Nabutovsky-Rotman 2003 [5]).

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Y. Liokumovich, Nabutovsky and Rotman (2017) [3]

Every Riemannian 2-sphere contains a simple closed geodesic

1. of length $\leq 5 \operatorname{diam} M$,
2. a second one of length $\leq 10 \operatorname{diam} M$,
3. and a third one of length $\leq 20 \operatorname{diam} M$.

Our Result

Proposition (B. 2023)

Given any point p in a (analytic) Riemannian 2-sphere M, there are two distinct simple geodesics loops based at p. One has length $\leq 7\,\mathrm{diam}(M)$ and the other has length $\leq 14\,\mathrm{diam}(M)$.

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Importantly, these two loops are simple/distinct. Without this restriction, the best known bound for a pair is

1. $\leq 4\,\text{diam}(M)$ (Rotman 2008 [6]), and
2. $\leq 12\,\text{diam}(M)$ (Cheng 2022 [1]).

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4. If the resulting geodesic loops are distinct, we are done. Otherwise, use standard homological techniques to find infinitely many short simple geodesic loops.

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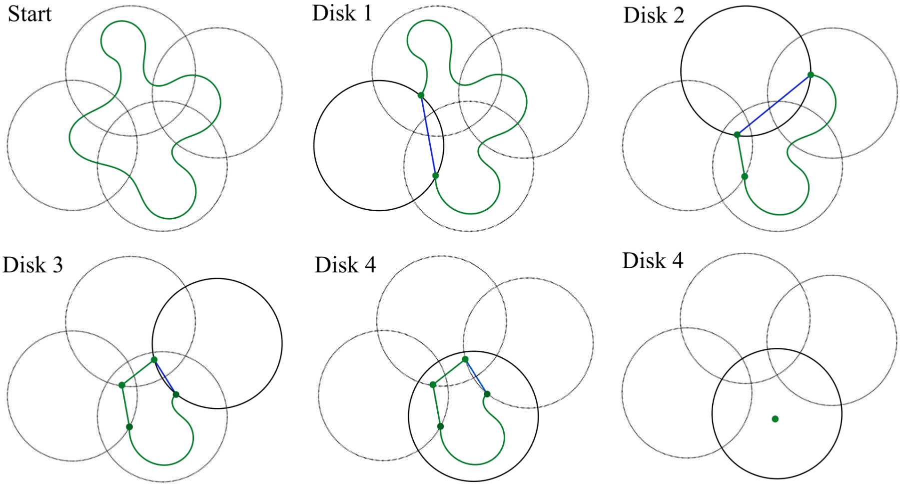
5. Repeat the entire above process until convergence.

Hass-Scott Disk Flow



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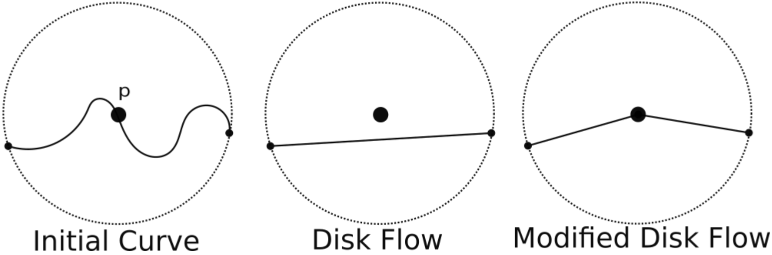
Our modified version of the disk flow will require all of these properties, except it will also have to ensure that loops at p remain based at p.

Modified Disk Flow: Definition

Suppose we want to shorten γ, a loop based at p.

Fixing the Basepoint

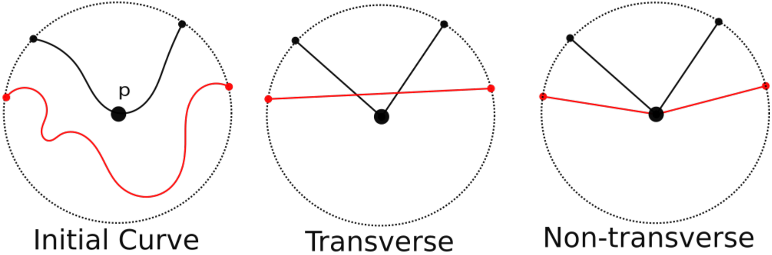
Fix γ at $t = 0$ by replacing the arc containing $p = \gamma(0)$ by the pair of geodesic rays that connect its endpoints to p.



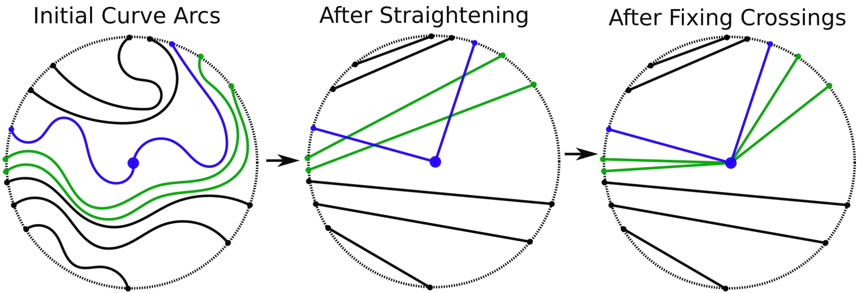
Modified Disk Flow: Definition

Preventing Intersections

If replacing an arc in B_1 by a minimizing geodesic would create transverse self-intersections, we instead replace it with the pair of geodesic rays that connect its endpoints to p.



Modified Disk Flow: Example



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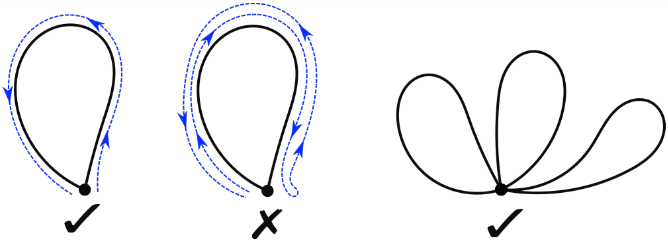
5. A modified disk flowed family is homotopic to the original family.

Modified Disk Flow: Convergence

Proposition (B. 2023)

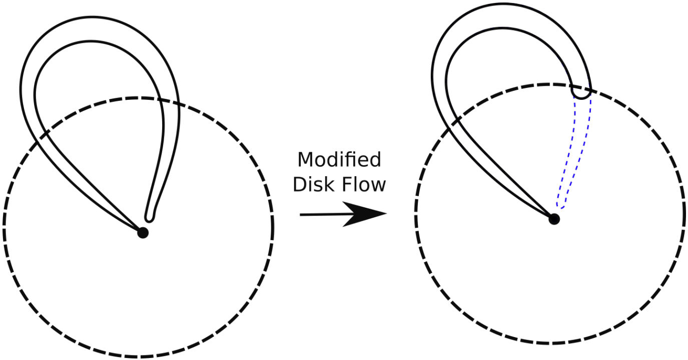
When we apply the modified disk flow to a simple loop based at p, a subsequence of the resulting curves converges to either

1. a point, or
2. a prime simple geodesic loop, or
3. a concatenation of simple geodesic loops, at least two of which have different images.



Modified Disk Flow: Convergence

For example, the flow cannot converge to a loop followed by its inverse:



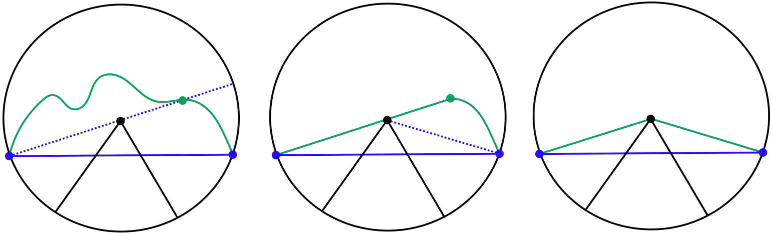
This means we will obtain two truly distinct loops when we flow our two families– not a pair of the form $\{\eta, \eta * -\eta\}$.

Modified Disk Flow: Shortening

Proposition (B. 2023)

The modified disk flow shortens curves.

We just need to check what happens when we replace an arc with two rays:



Constructing a Meridonial Sweepout

Next we use our new flow to create our sweepout.

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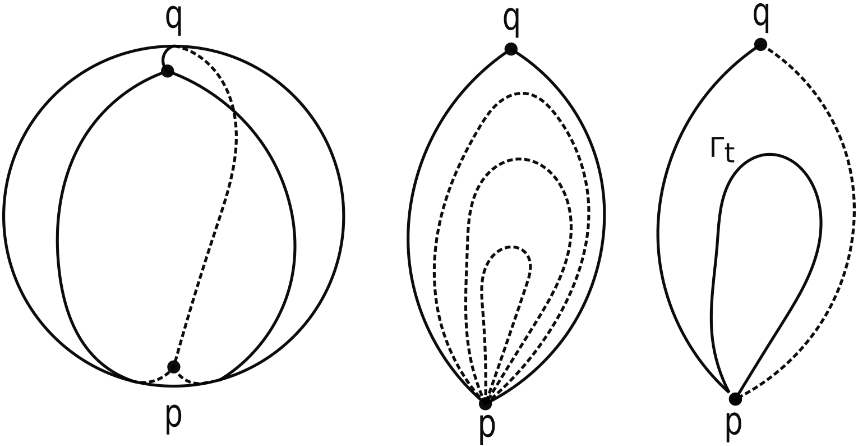
1. Using Berger's Lemma, divide M into regions bounded by minimizing geodesics connecting p to some point q.
2. Contract the boundary of each region through simple loops based at p using our flow (and some other tricks).
3. Convert these contractions into homotopies interpolating between the minimizing geodesics.

We get a family Γ_t of non-self-intersecting, non-mutually intersecting curves from p to q.

Constructing a Meridonial Sweepout

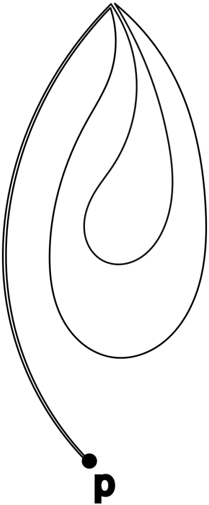


Pulling Tight

Apply the modified disk flow to the families $\Gamma_0 * -\Gamma_t$ and $\Gamma_s * -\Gamma_{s+t}$ to obtain two sets of simple geodesic loops.

If one set contains two distinct loops, we are done. Otherwise, we have obtained two prime loops.

If these geodesic loops are distinct, we are done. Otherwise, we can find an entire critical-level of geodesic loops using standard homological arguments.



Further Applications

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Can we find two distinct geodesic loops at a point p where the metric is degenerate?

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Seifert's Conjecture

Under certain conditions, on a domain D^n there are at least n (distinct) brake orbits with a given energy.

End

Thank you!

Section 1

References

[1] H. Y. Cheng. Curvature-free linear length bounds on geodesics in closed riemannian surfaces. *Transactions of the American Mathematical Society*, 375(7):5217–5237, 2022.

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