The Length of a Shortest Closed Geodesic

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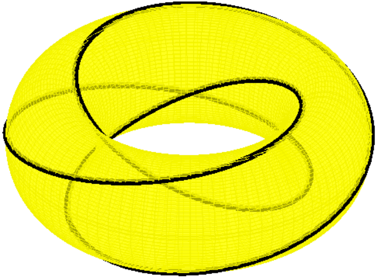
Background: Definitions

Definition

A geodesic is a curve that locally looks like a “straight line”. Alternatively, a curve that is locally length minimizing.

Examples:

- straight line segments in Euclidean space
- great circle arcs on the sphere
- this periodic curve on the torus



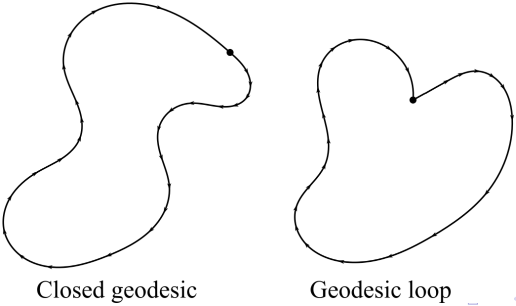
Background: Definitions

Definition

A geodesic that is also a closed curve is called a geodesic loop.

Definition

A geodesic loop that is smooth at its endpoints is called a closed (or periodic) geodesic.



Result

B. & R. Rotman (2020) [3]

Suppose M is a complete, orientable surface with finite area A and n ends. Let $l(M)$ be the length of a shortest closed geodesic on M.

1. If $n \leq 1$, then $l(M) \leq 4\sqrt{2A}$.
2. If $n \geq 2$, then $l(M) \leq 2\sqrt{2A}$.

Background: Existence

Question

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Every compact surface contains at least one closed geodesic.

G. Thorbergsson (1978) [13], V. Bangert (1980) [2]

Every complete surface of finite area contains at least one closed geodesic.

Background: Length Bounds

Question (M. Gromov)

What is the best bound for the length L of a shortest closed geodesic on a Riemannian manifold M in terms of $\sqrt[n]{\text{Vol}}$?

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What is the best bound for the length L of a shortest closed geodesic on a Riemannian manifold M in terms of $\sqrt[n]{\text{Vol}}$?

We consider $n = 2$. When M is compact and not a 2-sphere, answers in various cases were given by P. Pu, C. Loewner, M. Gromov, J. Hebda, Y. Burago, V. Zalgaller, and others.

Background: The Sphere

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Can we bound L if M is a sphere (with area A)?

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R. Rotman (2006) [10]

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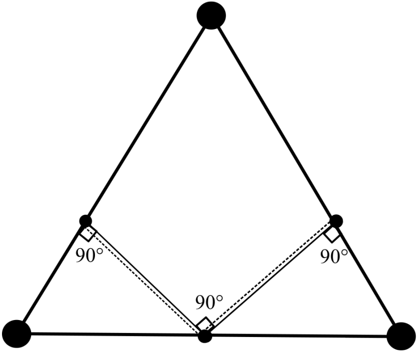
Conjecture (E. Calabi, Croke, & Gromov) [5]

The sharp bound is $L \leq 12^{1/4}\sqrt{A}$ and is realized by the Calabi-Croke sphere, obtained by gluing two equilateral triangles along their boundary.

Background: The Sphere



F. Balacheff (2010) [1], Sabourau (2010) [12]

This conjecture is true "locally", i.e. for metrics close to the Calabi-Croke sphere metric.

Background: Non-Compact Surfaces

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Suppose M is a complete, orientable surface with finite area A and n ends. Let $l(M)$ be the length of a shortest closed geodesic on M.

1. If $n = 1$, then $l(M) \leq 31\sqrt{A}$.
2. If $n = 2$, then $l(M) \leq (12 + 3\sqrt{2})\sqrt{A}$.
3. If $n \geq 3$, then $l(M) \leq 2\sqrt{2A}$.

Non-Compact Surfaces

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1. If $n \leq 1$, then $l(M) \leq 4\sqrt{2A}$.
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This is a tighter constant for $n = 1$ and $n = 2$.

Non-Compact Surfaces

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A. Jabbour, Sabourau (2020) [7] (arXiv)

1. This conjecture is true for $n = 3$.
2. For $n = 4$, the sharp bound is $L \leq (2/3^{1/4})\sqrt{A}$ and is realized by a tetrahedron with "cusps" on its vertices.

Curve Shortening

The Birkhoff Curve Shortening Process

Given any closed curve γ, we can produce a homotopy γ_t such that

1. $\gamma_0 = \gamma$
2. $L(\gamma_{t_2}) \leq L(\gamma_{t_1})$ for all $t_1 < t_2$,
3. γ_t either escapes to infinity, shrinks to a point, or converges (up to a subsequence) to a closed geodesic.

Curve Shortening

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Problem 2

When we shorten a curve, it might escape to infinity.

This is especially problematic if every curve is either nullhomotopic or homotopic to a point at infinity, i.e. if M is a sphere with $0, 1$, or 2 punctures.

Convexity

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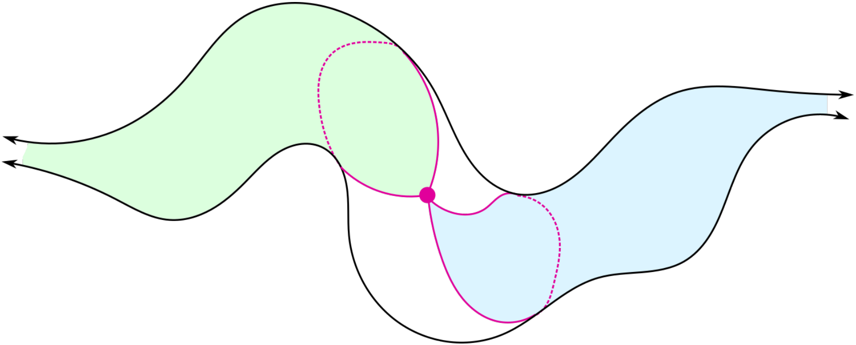
Lemma (cf. Croke (1988) [6])

Let Ω be a convex region. Let $\gamma \subset \overline{\Omega}$ be a closed curve and let γ_t be any curve in the homotopy produced by applying the Birkhoff curve shortening process to γ. Then $\gamma_t \subset \overline{\Omega}$.

"Curves cannot escape convex regions when shortened."

The Two-Ended Case

In this case, there always exists a pair of “short” geodesic loops that share a vertex but bound disjoint, cylindrical, **convex** regions.

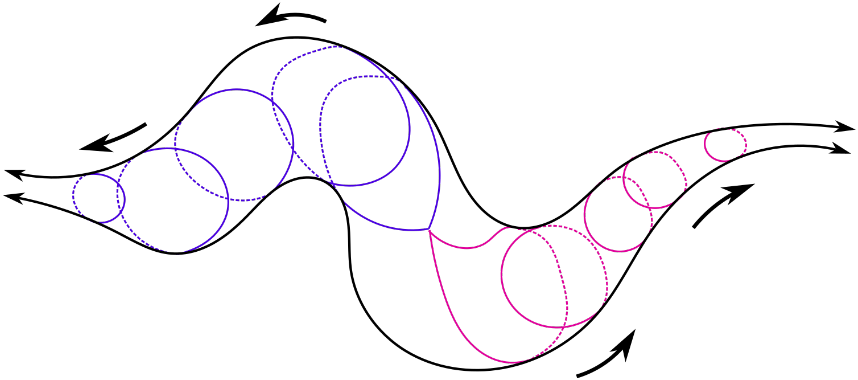


The Two-Ended Case

First Idea

Shorten each loop individually.

Either both loops will escape to infinity or we get a short closed geodesics.

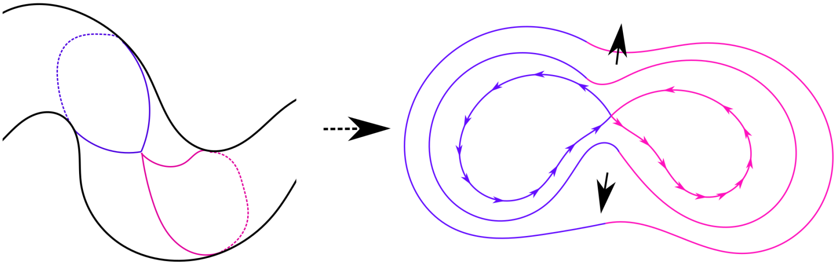


The Two-Ended Case

Second Idea

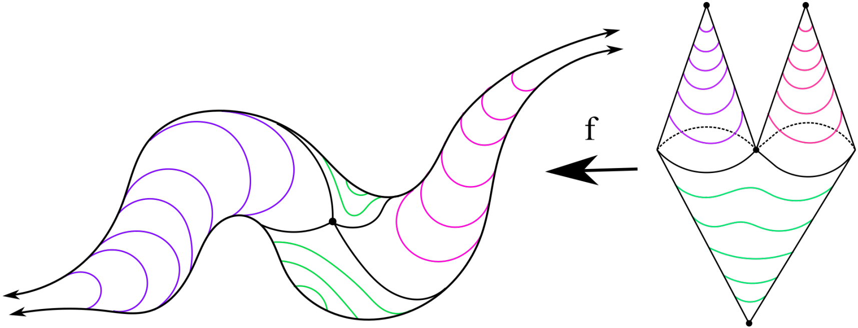
Shorten the loop pair as a single curve.

Either the curve will contract to a point or we get a short closed geodesic.



The Two-Ended Case

If we still haven't found a closed geodesic, then we have covered our entire surface with homotopies of curves.
Combine these three homotopies to make a sphere map f of non-zero degree.



The Two-Ended Case

Gromov's Idea: Pseudo-extension

Any attempt to continuously extend a map $f : S^2 \to S^2$ of non-zero degree to some $\hat{f} : B^2 \to S^2$ is doomed to fail, because S^2 is not contractible.

If we try to construct $\hat{f}$ very carefully, this failure can tell us something about f.

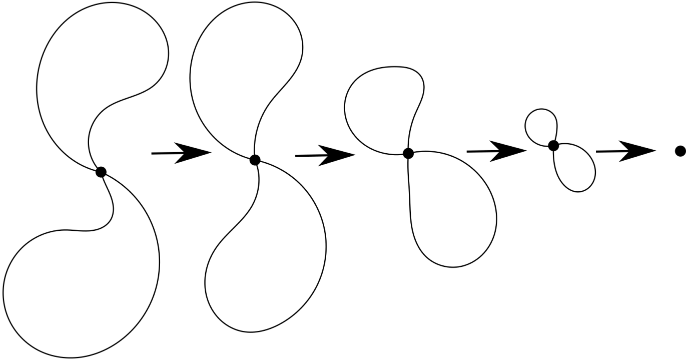
We will now try (in vain) to extend our map f.

Constructing the Pseudo-extension

Third Idea

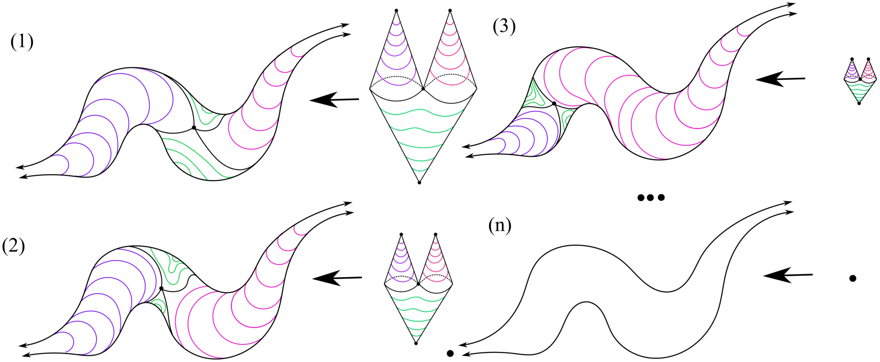
Shorten the loop pair as a *geodesic net*, i.e. a graph with one vertex and two edges.

Critical fact: this loop pair will either contract to a point or converge to a figure-eight closed geodesic.



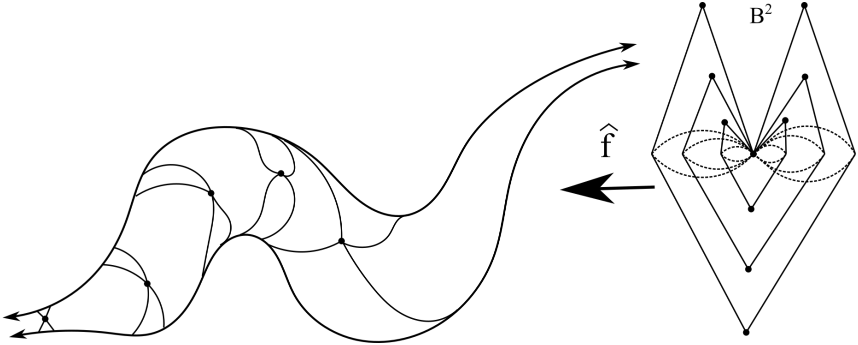
Constructing the Pseudo-extension

Supposing we don't get a geodesic, we make a (possibly zero-degree) sphere map for each loop pair in our net shortening homotopy, until our net becomes a point.



Constructing the Pseudo-extension

This creates an impossible continuous extension of f to the solid ball.



Therefore we must have encountered a short closed geodesic at some point. □

Further Applications of Curve Shortening

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What is the optimal constant...

- ...for $l(M)$ in S^2?
- ...for $l(M)$ in a non-compact surface (of finite area)?

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Question

What about geodesics in higher dimensional manifolds? What about minimal hypersurfaces?

Conclusion

Thank you!

Section 1

References

[1] F. Balacheff. A local optimal diastolic inequality on the two-sphere. *Journal of Topology and Analysis*, 02(01): 109–121, 2010.

[2] V. Bangert. Closed geodesics on complete surfaces. *Mathematische Annalen*, 251(1):83–96, 1980.

[3] I. Beach and R. Rotman. The length of the shortest closed geodesic on a surface of finite area. *Proceedings Of The American Mathematical Society*, 2020.

[4] G. Birkhoff. *Dynamical Systems*. American Mathematical Society. American Mathematical Society, 1927. ISBN 9780821810095.

[5] E. Calabi and J. G. Cao. Simple closed geodesics on convex surfaces. *Journal of Differential Geometry*, 36(3):517–549, 1992.

[6] C. B. Croke. Area and the length of the shortest closed geodesic. *J. Differential Geom.*, 27(1):1–21, 1988.

[7] A. Jabbour and S. Sabourau. Sharp upper bounds on the length of the shortest closed geodesic on complete punctured spheres of finite area. 2020. Preprint.

[8] Y. Liokumovich, A. Nabutovsky, and R. Rotman. Lengths of three simple periodic geodesics on a Riemannian 2-sphere. *Mathematische Annalen*, 367:831–855, 2017.

[9] A. Nabutovsky and R. Rotman. The length of the shortest closed geodesic on a 2-dimensional sphere. *International Mathematics Research Notices*, 2002(23):1211–1222, 2002.

[10] R. Rotman. The length of a shortest closed geodesic and the area of a 2-dimensional sphere. *Proceedings of the American Mathematical Society*, 134(10):3041–3047, 2006.

[11] S. Sabourau. Filling radius and short closed geodesics of the 2-sphere. *Bulletin de la Société Mathématique de France*, 132(1):105–136, 2004.

[12] S. Sabourau. Local extremality of the Calabi–Croke sphere for the length of the shortest closed geodesic. *Journal of the London Mathematical Society*, 82(3):549–562, 2010.

[13] G. Thorbergsson. Closed geodesics on non-compact Riemannian manifolds. *Mathematische Zeitschrift*, 159(3): 249–258, 1978.