Geodesics and Curve Shortening

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Section 1

Geodesics

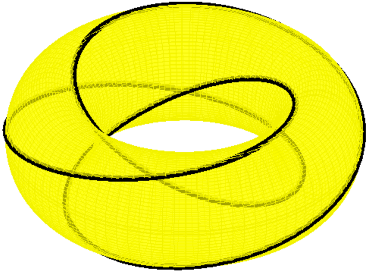
Background: Geodesics

Definition

A geodesic is a curve that locally looks like a "straight line". Alternatively, a curve that is locally length minimizing.

Examples:

- straight line segments in Euclidean space
- great circle arcs on the sphere
- this periodic curve on the torus



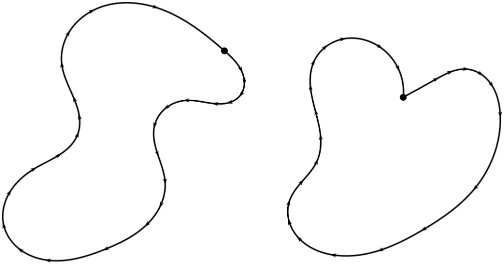
Background: Geodesics

Definition

A geodesic that is also a closed curve is called a geodesic loop.

Definition

A geodesic loop that is smooth at its endpoints is called a closed (or periodic) geodesic.



Closed geodesic   Geodesic loop

Background: Existence

Question

Does a closed geodesic always exist on a given surface?

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Every compact surface contains at least one closed geodesic.

G. Thorbergsson (1978) [16], V. Bangert (1980) [2]

Every complete surface of finite area contains at least one closed geodesic.

Background: Length Bounds

Question (M. Gromov)

What is the best bound for the length L of a shortest closed geodesic on a Riemannian manifold M in terms of its geometric properties (e.g., $\sqrt[n]{\mathrm{Vol}}$, diameter, filling radius)?

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What is the best bound for the length L of a shortest closed geodesic on a Riemannian manifold M in terms of its geometric properties (e.g., $\sqrt[n]{\mathrm{Vol}}$, diameter, filling radius)?

We consider $n = 2$. When M is compact and not a 2-sphere, answers in various cases were given by P. Pu, C. Loewner, M. Gromov, J. Hebda, Y. Burago, V. Zalgaller, and others.

Background: The Sphere

Question

Can we bound L if M is a sphere (with area A)?

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Conjecture (E. Calabi, Croke, & Gromov) [5]

The sharp bound is $L \leq 12^{1/4}\sqrt{A}$ and is realized by the Calabi-Croke sphere, obtained by gluing two equilateral triangles along their boundary.

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F. Balacheff (2010) [1], Sabourau (2010) [15]

This conjecture is true "locally", i.e. for metrics close to the Calabi-Croke sphere metric.

Background: Non-Compact Surfaces

Question

Can we bound L if M is non-compact (with area A)?

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Can we bound L if M is non-compact (with area A)?

Croke (1988) [7]

Suppose M is a complete, orientable surface with finite area A and n ends. Let $l(M)$ be the length of a shortest closed geodesic on M.

1. If $n = 1$, then $l(M) \leq 31\sqrt{A}$.
2. If $n = 2$, then $l(M) \leq (12 + 3\sqrt{2})\sqrt{A}$.
3. If $n \geq 3$, then $l(M) \leq 2\sqrt{2A}$.

Non-Compact Surfaces

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1. If $n \leq 1$, then $l(M) \leq 4\sqrt{2A}$.
2. If $n \geq 2$, then $l(M) \leq 2\sqrt{2A}$.

This is a sharper constant for $n = 1$ and $n = 2$.

Non-Compact Surfaces

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What is the sharpest possible bound for L if M is non-compact (with area A)?

Conjecture (B. & Rotman (2019)) [3]

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Non-Compact Surfaces

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Sabourau-Jabbour (2020) [10]

1. This conjecture is true for $n = 3$.
2. For $n = 4$, the sharp bound is $L \leq (2/3^{1/4})\sqrt{A}$ and is realized by a tetrahedron with "cusps" on its vertices.

Question

What techniques can we use to analyze closed geodesics on a surface?

Section 2

Our First Tool: Curve Shortening Algorithms

The Birkhoff Curve Shortening Process

Idea

We can find short geodesics by shortening short closed curves.

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The Birkhoff Curve Shortening Process

Given any closed curve γ on a compact manifold, we can produce a homotopy γ_t such that

1. $\gamma_0 = \gamma$
2. $L(\gamma_{t_2}) \leq L(\gamma_{t_1})$ for all $t_1 < t_2$,
3. γ_t either escapes to infinity, shrinks to a point, or converges on a closed geodesic.

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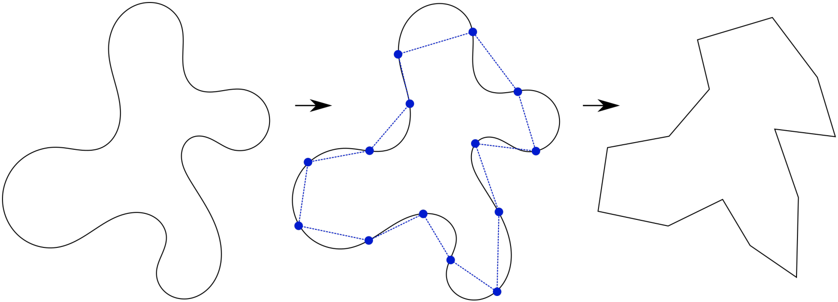
3. Form a new curve by connecting $\gamma(t_i)$ to $\gamma(t_{i+1})$ by the corresponding unique minimizing geodesic segment.
4. Homotope γ to this new curve.
5. Repeat the above process, while ensuring that the new chosen points $\gamma(t_i')$ do not coincide with the previous chosen points $\gamma(t_i)$.

Example: BCSP in the Plane



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When I shorten a curve, it might collapse to a point (i.e., a trivial closed geodesic).

Problem 2

When I shorten a curve, it might escape to infinity.

This is bad if every curve is either nullhomotopic or homotopic to a point at infinity, i.e. if M is a sphere with punctures.

Convexity

One way to control how curves shorten is to "trap" them in convex regions.

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Definition

A connected region Ω is called convex if there is some $\epsilon > 0$ such that for all $x, y \in \overline{\Omega}$ with $d(x, y) < \epsilon$, the minimizing geodesic segment between x and y lies within $\overline{\Omega}$.

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Lemma

Let γ be a curve bounding a region Ω.

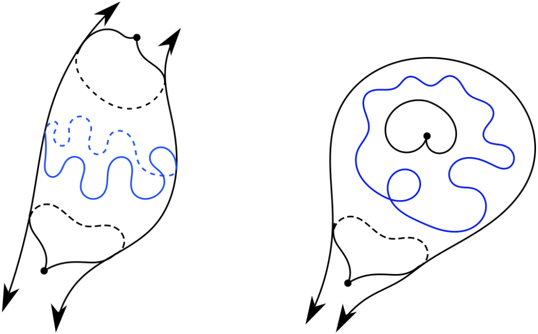
1. If γ is a geodesic, Ω is convex.
2. If γ is a geodesic loop and its inward-facing angle is less than π, Ω is convex.

Convexity

Lemma

Let Ω be a convex region. Let $\gamma \subset \overline{\Omega}$ be a closed curve and let γ_t be any curve in the homotopy produced by applying the Birkhoff curve shortening process to γ. Then $\gamma_t \subset \overline{\Omega}$.

"Curves cannot escape convex regions when shortened."



Convexity

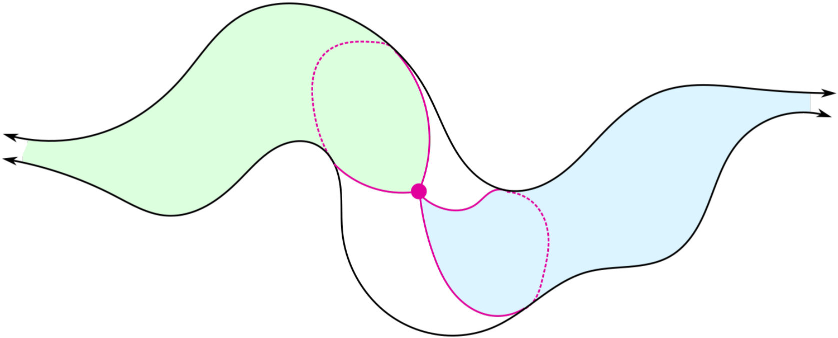
Non-compact surfaces with finite area have lots of geodesic loops that we can exploit to control our curve shortening flow.

In fact, every infinite end is contained in a convex set bounded by a "short" geodesic loop.

We will illustrate the case when M is a sphere with two ends ($M \simeq S^1 \times \mathbb{R}$).

The Two-Ended Case

In this case, there always exists a pair of “short” geodesic loops that share a vertex but bound disjoint, cylindrical, **convex** regions.

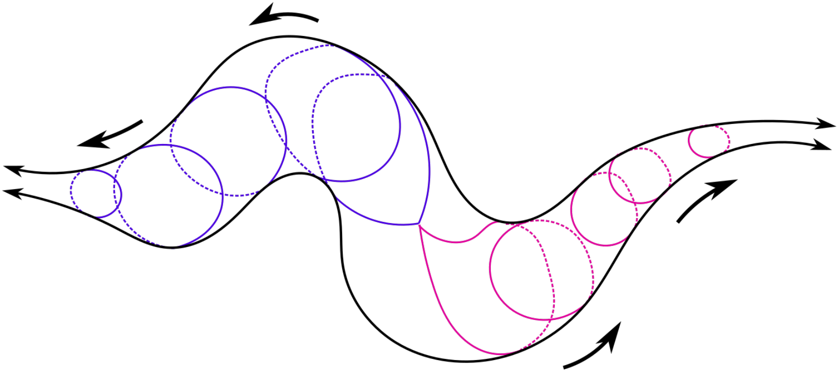


The Two-Ended Case

First Idea

Shorten each loop individually.

Either both loops will escape to infinity or we get a short closed geodesics.

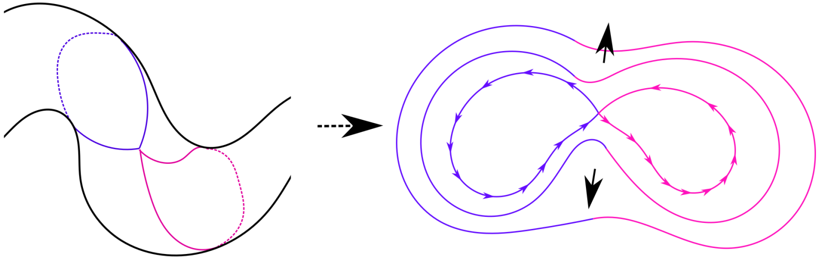


The Two-Ended Case

Second Idea

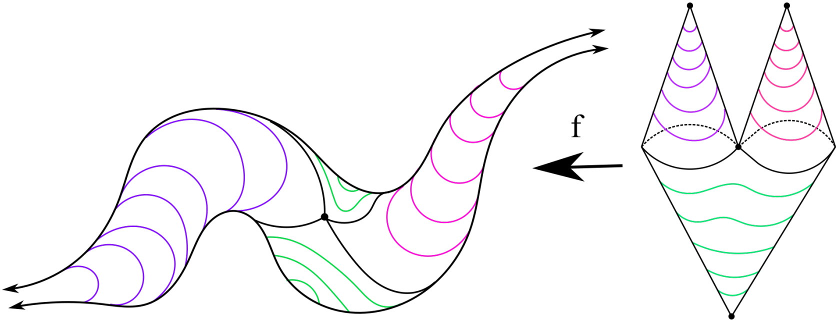
Shorten the loop pair as a single curve.

Either the loop will contract to a point or we get a short closed geodesic.



The Two-Ended Case

If we still haven't found a closed geodesic, then we have covered our entire surface with homotopies of curves. Combine these three homotopies to make a sphere map f of non-zero degree.



The Two-Ended Case

Gromov's Idea: Pseudo-extension

Any attempt to continuously extend a map $f : S^2 \to S^2$ of non-zero degree to some $\hat{f} : B^2 \to S^2$ is doomed to fail, because S^2 is not contractible.

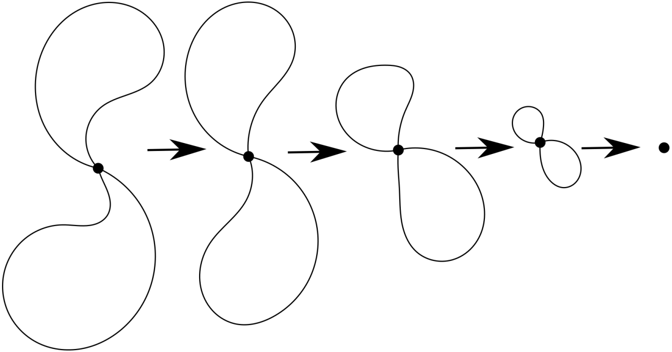
We will now try (in vain) to extend our map f.

Constructing the Pseudo-extension

Third Idea

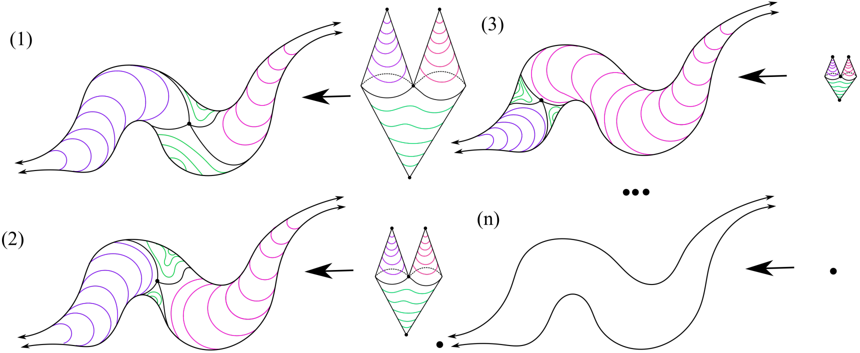
Shorten the loop pair as a *geodesic net*, i.e. a graph with one vertex and two edges.

Critical fact: this loop pair will either contract to a point or converge to a figure-eight closed geodesic.



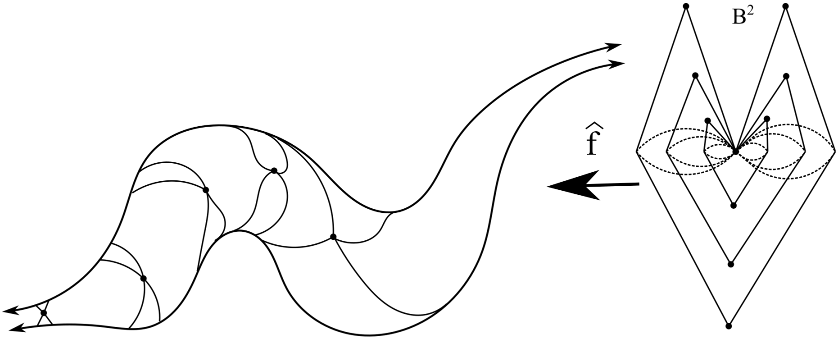
Constructing the Pseudo-extension

Supposing we don't get a geodesic, we make a (possibly zero-degree) sphere map for each loop pair in our net shortening homotopy, until our net becomes a point.



Constructing the Pseudo-extension

This creates an impossible continuous extension of f to the solid ball.



Therefore we must have encountered a short closed geodesic at some point. $\square$

Question

What other contexts can we apply these techniques to?

Section 3

Finding Short Closed Geodesics in a Degenerate Metric

Hamiltonian Systems

Consider on $\mathbb{R}^n \times \mathbb{R}^n$ the Hamiltonian

$$
\begin{aligned}
H(p, q) &= \frac{1}{2} p^T A(q) p + V(q) \\
A &: \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \\
V &: \mathbb{R}^n \to \mathbb{R}
\end{aligned}
$$

with corresponding Hamiltonian system

$$
\begin{aligned}
\dot{p} &= -\frac{\partial H}{\partial q} = -\frac{1}{2} p^T \frac{\partial A(q)}{\partial q} p - \frac{\partial V(q)}{\partial q} \\
\dot{q} &= \frac{\partial H}{\partial p} = A(q) p
\end{aligned}
$$

Hamiltonian Systems

Suppose instead our domain is a Riemannian manifold with metric given by the matrix $\frac{1}{2}A(q)$. Then our Hamiltonian system is equivalent to the equation

$$\frac{D}{dt}\dot{q} + \nabla V(q) = 0$$

(note that p is entirely determined by q)

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Question

Are there periodic solutions to this equation? What do they look like?

Brake Orbits

Because the Hamiltonian is constant along solution curves, there is a constant E so that

$$\frac{1}{2}\|\dot{q}\|^2 + V(q) = E$$

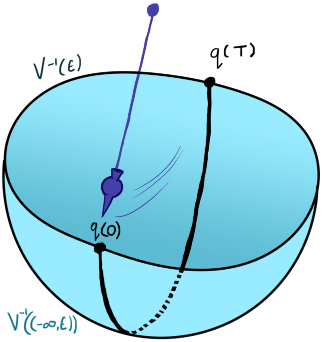
This is the "energy" of the solution.

Therefore a solution q must lie within $\Omega_E = V^{-1}((-\infty, E])$, and $\dot{q} = 0$ only on the set $V^{-1}(E)$.

Brake Orbits

Definition

A brake orbit is a solution curve $t \mapsto (p(t), q(t))$ with $p(0) = p(T) = 0$ for some T. Necessarily we have $q(0), q(T) \in V^{-1}(E)$.



Seifert's Conjecture

If E is a regular value of V and $V^{-1}((-\infty, E])$ is homeomorphic to a disk D^n, then our system has n distinct brake orbit solutions.

Question

How can we find brake orbits?

The Maupertuis Principle

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Solutions of our system in (M, g) with energy E correspond to geodesics $q(t)$ in the metric $g_E(x) = (E - V(x))g(x)$.

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Problem

It is difficult to use this principle to find brake orbits because the metric g_E is degenerate (zero) on $V^{-1}(E)$.

Orthogonal Geodesic Chords

One possible strategy: orthogonal geodesic chords.

Orthogonal Geodesic Chords

Pick "small" δ and consider the set $\Omega_\delta = V^{-1}((-\infty, E - \delta])$ with the metric g_E. We call a geodesic segment γ an orthogonal geodesic chord if

- The endpoints of γ lie on $\partial\Omega_\delta$.
- γ is orthogonal to $\partial\Omega_\delta$ at its endpoints.
- γ does not intersect $\partial\Omega_\delta$ except at its endpoints.

Orthogonal Geodesic Chords

Giambó-Giannoni-Piccione 2022 [8]

- In the metric g_E, there is some $\epsilon > 0$ such that for any $0 < \delta < \epsilon$ an orthogonal geodesic chord in Ω_δ can be uniquely extended to a break orbit in Ω_E.
- In the metric g_E in n dimensions, there are n distinct orthogonal geodesic chords.

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- In the metric g_E in n dimensions, there are n distinct orthogonal geodesic chords.

Proof:

- Use a specialized curve shortening flow.
- Count the fixed points of the flow using Lyusternik-Schnirelman theory.

Question

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Can we apply our curve shortening techniques to find two brake orbits of bounded length in a 2-disk Ω_E?

Problems

Problem

Geodesics that pass through $\partial\Omega_E$ are not unique and do not have to be locally-length minimizing.

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Solution

There is still a number $r > 0$ such that any two points of distance at most r can be connected by a unique minimizing geodesic.

Problems

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If we want to obtain two distinct brake orbits, we will need to ensure that when we shorten curves we do not obtain a pair of the form $\{\gamma, \gamma^2\}$.

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Solution

Look for simple curves.

Overview

1. Find two sweepouts of Ω_E through curves based at $p = \partial\Omega_E$ of bounded length.

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2. Modify these sweepouts to be through simple curves of bounded length.
3. Apply a based-loop length-shortening process to these sweepouts– without introducing self-intersections– to obtain two simple geodesic loops based at $p = \partial\Omega_E$ of bounded length.

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4. If these geodesic loops are distinct, we are done. Otherwise, use Morse theory to find an entire critical-level of geodesic loops.

Constructing Sweepouts of a Sphere

Step one: find two sweepouts of Ω_E through curves of known length. For example, we could use the following.

Liokumovich-Nabutovsky-Rotman (2015) [11]

Given a 2-dimensional disk D and any point $q \in \partial D$, there is a sweepout γ_t of D through loops based at q with $\gamma_0 = q$, $\gamma_1 = \partial D$ and

$$L(\gamma_t) \leq 2L(\partial D) + 664\sqrt{\text{area}(D)} + 2\text{diam}(D)$$

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Our first sweepout will be the above, and the second will be $\Gamma(s,t) = \gamma_t * \gamma_s$.

Constructing a Monotone Sweepout

Step two: modify these sweepouts to be through simple curves.

Chambers-Liokumovich (2014) [6]

Given a sweepout of a 2-sphere M through curves of length at most L, it is possible to construct a sweepout of M through **simple** curves of length at most L.

Disk Flow

Step three: how do we shorten loops without introducing intersections? Our starting point is the following process.

Hass-Scott (1994) [9]

Given any closed curve γ on a compact manifold, we can produce a homotopy γ_t such that

1. $\gamma_0 = \gamma$
2. $L(\gamma_{i+1}) \leq L(\gamma_i)$ for all $i \in \mathbb{N}$,
3. A subsequence of the curves γ_i either shrinks to a point or converges on a closed geodesic.
4. **The number of self-intersections of γ_i does not increase.**

Disk Flow

1. Cover the manifold with a finite collection of metric balls $\{B_i\}_{i=1}^n$ of radius $r < \text{inj}(M)/2$. Ensure that γ is not tangent to any ∂B_i.

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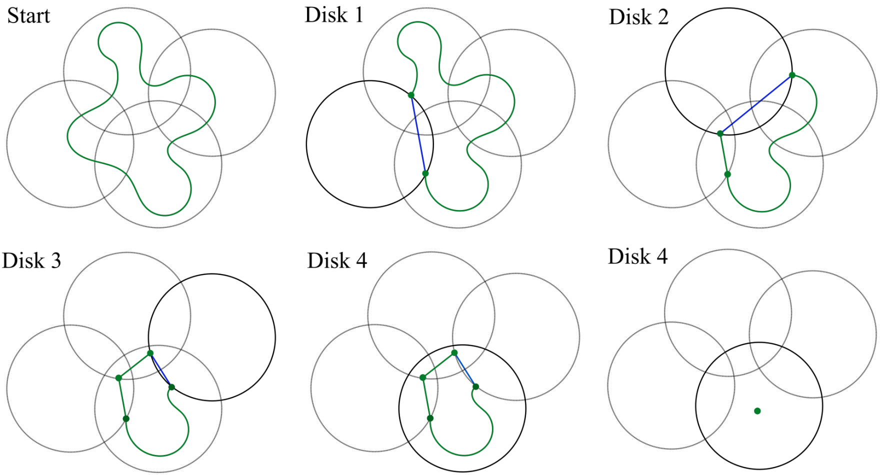
3. Homotope γ to this new curve without introducing new self-intersections.
4. Repeat this process with $B_2, \ldots, B_n$.
5. Repeat the entire above process until convergence.

Disk Flow

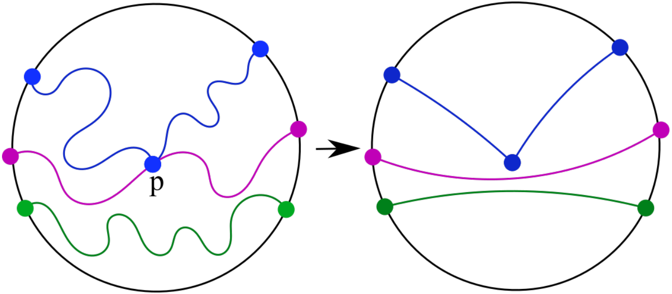


Modified Disk Flow

First Modification

Ensure that B_1 is centred on $p = \partial\Omega_E$. At each step in B_1, we will replace the arc that contains the basepoint $\gamma_t(0)$ with two minimizing rays emanating from $\gamma_t(0)$.

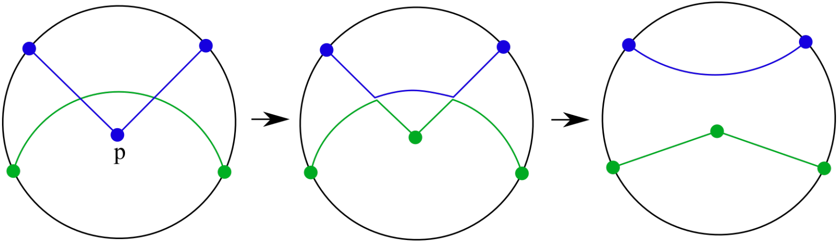
This ensures that the basepoint remains fixed under the flow.



Modified Disk Flow

Second Modification

It is possible for a minimizing geodesic to intersect the two minimizing rays, even if the original curve was simple. When such an intersection occurs, we apply the following homotopy.



Modified Disk Flow

Proposition (B. (2022))

Given any closed, **simple** curve γ with basepoint $\partial\Omega_E$, we can produce a homotopy γ_t of loops based at $\partial\Omega_E$ such that

1. $\gamma_0 = \gamma$
2. $L(\gamma_{i+1}) \leq L(\gamma_i)$ for all $i \in \mathbb{N}$,
3. A subsequence of the curves γ_t either shrinks to a point or converges on a **(locally length-minimizing)** closed geodesic γ_∞.
4. **Each γ_t only intersects itself at p, and it only does so non-transversely.**
5. **The loop γ_∞ is prime (i.e., it is not given by iterating another loop).**

Modified Disk Flow

Important Fact

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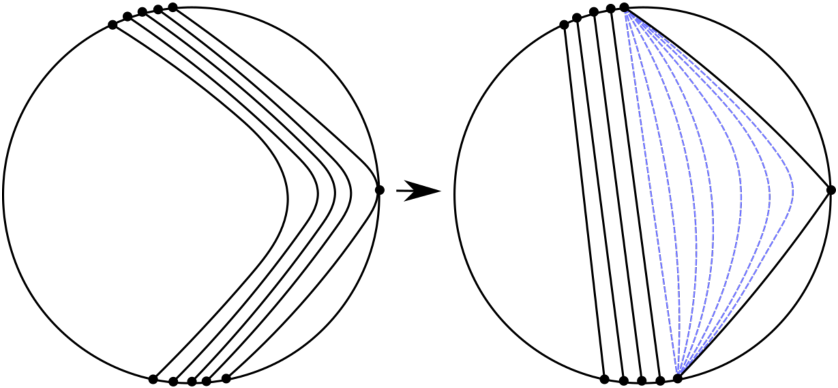
2. Because each only intersects itself non-transversely at p, the only possibility is $k = 1$ and hence $\gamma_\infty = \eta$ is prime.

Extending the Disk Flow to Sweepouts

For continuous families of curves, we cannot ensure that there are no curve tangent to the boundary of a disk. If a tangency occurs, we fill in the "gaps" with a homotopy.



Pulling Tight

We finish by applying the curve shortening flow to our two sweepouts. We cannot obtain an iterated loop, so either the two loops have distinct images or they are equal.

If these geodesic loops are distinct, we are done. Otherwise, use Morse theory to find an entire critical-level of geodesic loops.

Result

Combining everything, we have:

Proposition (B. (2022))

The space (Ω_E, g_E) has two distinct geodesic loops based at $p = \partial\Omega_E$ of bounded length, e.g., with

$$L \leq 1328\sqrt{\text{area}(\Omega_E)} + 4\text{diam}(\Omega_E)$$

Consequently, the associated Hamiltonian system on (M, g) has two distinct brake orbits of energy E with bounded length.

Section 4

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