# The Length of a Shortest Closed Geodesic

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# Geodesics

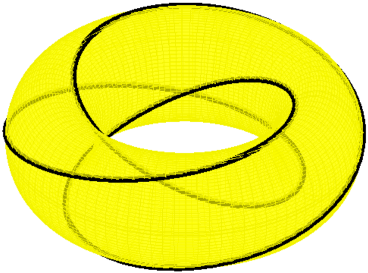
## Definition

A geodesic is a curve that locally looks like a "straight line".

Alternatively, a curve that is locally length minimizing.

Examples:

- straight line segments in Euclidean space
- great circle arcs on the sphere
- this periodic curve on the torus



# Geodesics

## Definition

A geodesic that is a closed curve is called a geodesic loop.

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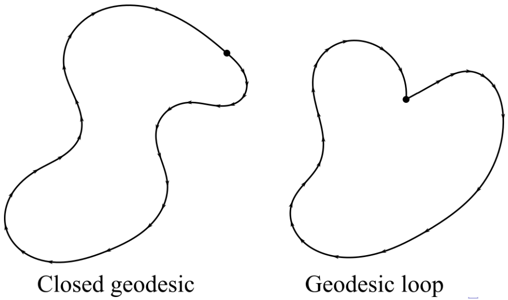
A geodesic loop that is smooth at its endpoints is called a closed (or periodic) geodesic.



# Background: Existence

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### G. Birkhoff (1917)

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### G. Thorbergsson (1978) [9], V. Bangert (1980) [1]

Every complete surface of finite area contains at least one closed geodesic.

# Background: Length Bounds

### Question (M. Gromov)

What is the best bound for the length $L$ of a shortest closed geodesic on a Riemannian manifold $M$ in terms of its geometric properties (e.g., $\sqrt[n]{\text{Vol}}$, diameter, filling radius)?

# Background: Length Bounds

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What is the best bound for the length $L$ of a shortest closed geodesic on a Riemannian manifold $M$ in terms of its geometric properties (e.g., $\sqrt[n]{\text{Vol}}$, diameter, filling radius)?

When $M$ is compact surface except a 2-sphere, answers in various cases were given by P. Pu, C. Loewner, M. Gromov, J. Hebda, Y. Burago, V. Zalgaller, and many others.

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# Background: The Sphere

## Conjecture (E. Calabi, Croke, & Gromov)

The sharp bound is $L \leq 12^{1/4}\sqrt{A}$ and is realized by the Calabi-Croke sphere, obtained by gluing two equilateral triangles along their boundary.

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## F. Balacheff (2010), Sabourau (2010) [8]

This conjecture is true "locally", i.e. for metrics close to the Calabi-Croke sphere metric.

# Background: Non-Compact Surfaces

## Theorem (Croke, 1988) [2]

Suppose $M$ is a complete, orientable surface with finite area $A$ and $n$ ends. Let $L$ be the length of a shortest closed geodesic on $M$.

1. If $n = 1$, then $L \leq 31\sqrt{A}$.
2. If $n = 2$, then $L \leq (12 + 3\sqrt{2})\sqrt{A}$.
3. If $n \geq 3$, then $L \leq 2\sqrt{2A}$.

# Main Result

**Theorem (I.B. & R. Rotman 2020)**

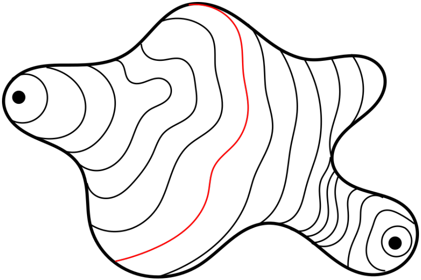
Suppose $M$ is a complete, orientable surface with finite area $A$ and $n$ ends. Let $L$ be the length of a shortest closed geodesic on $M$.

1. If $n \leq 1$, then $L \leq 4\sqrt{2A}$.
2. If $n \geq 2$, then $L \leq 2\sqrt{2A}$.

This is an improvement of Croke's result if $n = 1$ or 2. We will illustrate the (easier) case $n = 2$.

# Starting Point: Birkhoff's Method

Let $M$ be a sphere. Suppose we can construct a family of loops starting and ending at a point curve that induces a non-zero degree sphere map.



Then $M$ has a geodesic with length bounded by the length of the longest loop in our family.

# Difficulty

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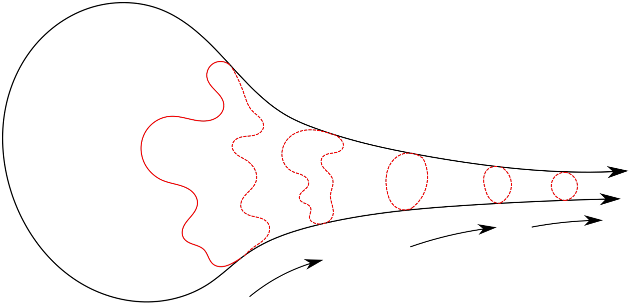
However, if we try to do this on a *punctured* sphere, the curve might escape to infinity.



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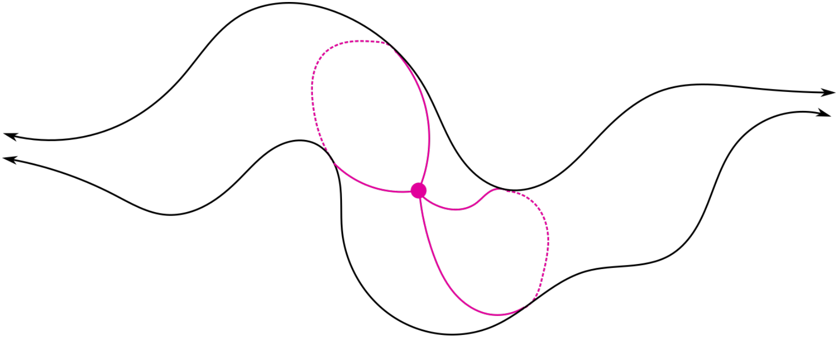
## Gromov's Idea

Any attempt to continuously extend a map $f : S^2 \to S^2$ of non-zero degree to some $\hat{f} : B^2 \to S^2$ is doomed to fail, because $S^2$ is not contractible.

If we try to construct $\hat{f}$ in a smart way, this failure can tell us some information.

# The Two-Ended Case

If $M \simeq S^1 \times \mathbb{R}$, there always exists a pair of "short" geodesic loops that share a vertex but bound disjoint cylindrical regions.

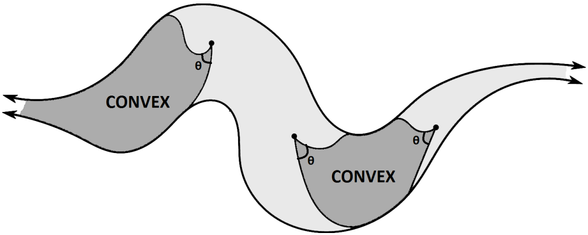


Assuming that our manifold has no short closed geodesics, we will use this structure to construct $f$.

# Convex Sets

**Lemma**

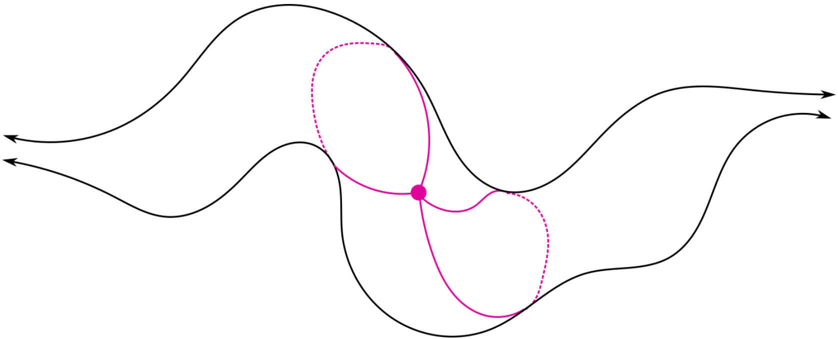
Suppose a region $\Omega$ is bounded by geodesic loops. If each geodesic loop has inward-facing angle at most $\pi$, then $\Omega$ is "convex".




Importantly, a curve cannot leave a convex set when being shortened.

# The Two-Ended Case: Convexity

In fact, our two special loops bound disjoint *convex* regions.

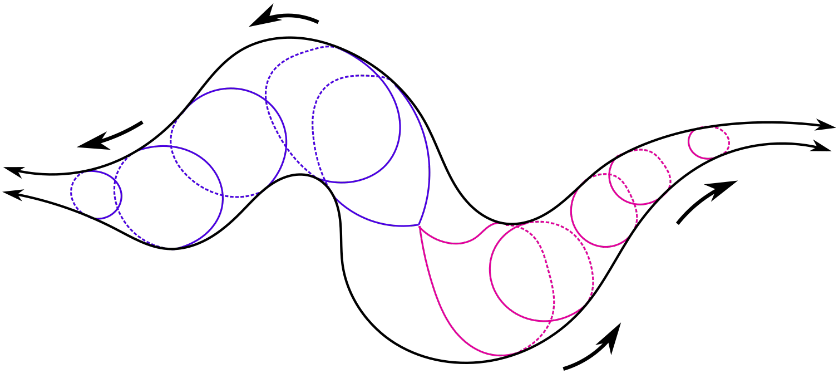


# Constructing the Sphere Map

## First Step

Shorten each loop individually.

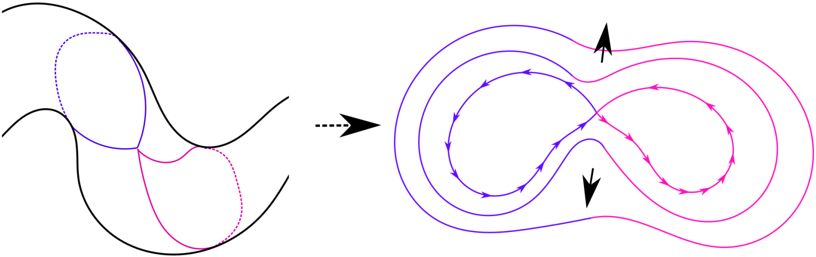
Both loops will escape to infinity because there are no short closed geodesics.



# Constructing the Sphere Map

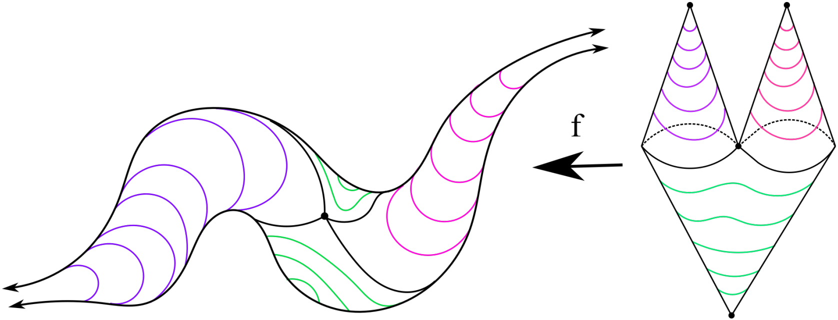
## Second Step

Shorten the loop pair as a single curve.



# Constructing the Sphere Map

Combine these three homotopies to make a sphere map $f$ of non-zero degree (by convexity & absence of geodesics).

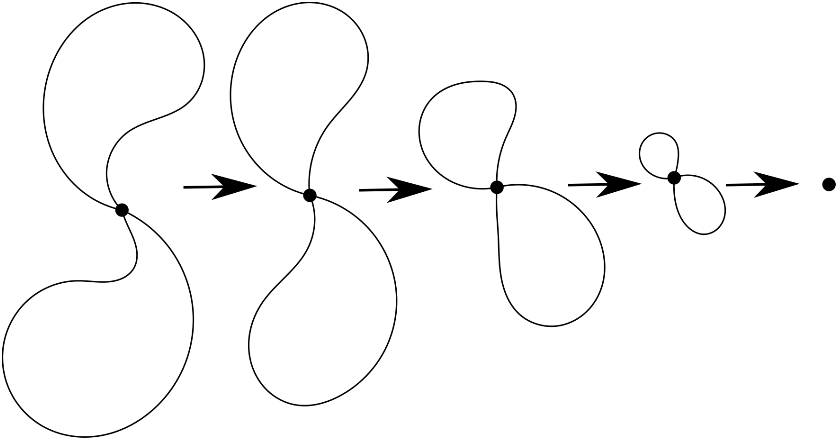



We will now try to extend this map.

# Constructing the Pseudo-extension

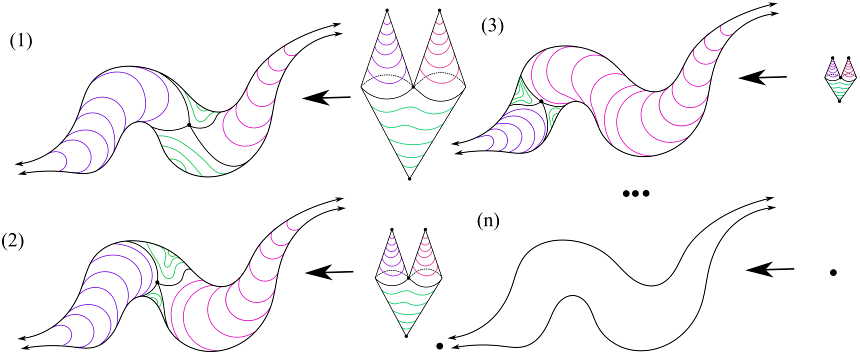
## Third Step

Shorten the loop pair as a *geodesic net*, i.e. a graph with one vertex and two edges.



# Constructing the Pseudo-extension

We get a (possibly zero-degree) sphere map for each loop pair in our net shortening homotopy, until our net becomes a point.



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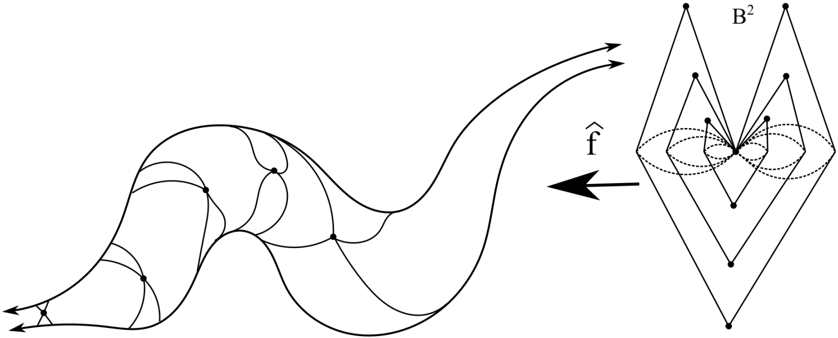
This creates an impossible continuous extension of $f$ to the solid ball.



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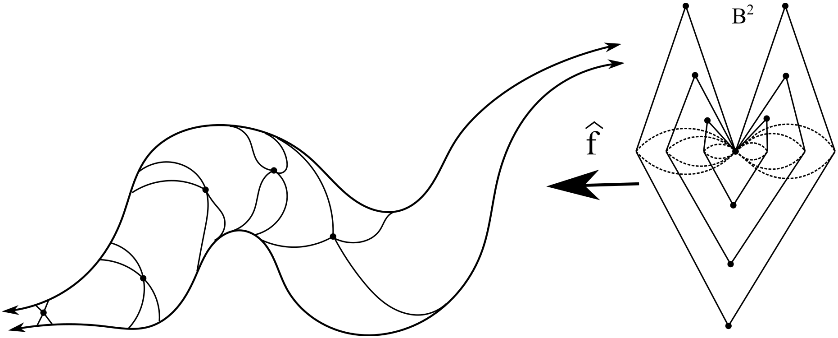


Our assumption that there are no short geodesics is false! □

# Further Questions

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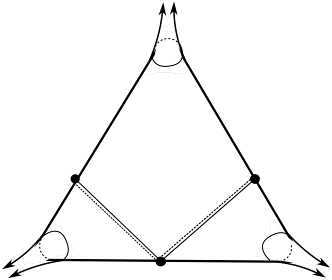
Conjecture (B.-Rotman):
If $M$ is a sphere with $n \leq 3$ punctures, $12^{1/4}\sqrt{A}$ is the sharp bound and is realized by the Calabi-Croke sphere with "cusps" on its vertices.



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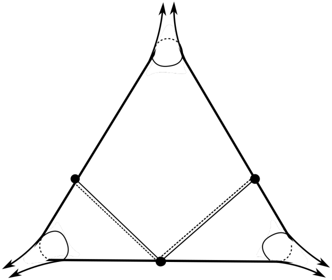

This was recently proven to be true for $n = 3$ by A. Jabbour and S. Sabourau [3].

# Further Questions

### Question 2

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Y. Liokumovich, Nabutovsky and Rotman (2017) [4]:
Every Riemannian 2-sphere contains a simple closed geodesic of length $\leq 5\,\mathrm{diam}\, M$, a second one of length $\leq 10\,\mathrm{diam}\, M$ and a third one of length $\leq 20\,\mathrm{diam}\, M$.

# Further Questions

## Question 3

What about higher dimensions?

[1] V. Bangert. Closed geodesics on complete surfaces. *Mathematische Annalen*, 251(1):83–96, 1980.

[2] C. B. Croke. Area and the length of the shortest closed geodesic. *J. Differential Geom.*, 27(1):1–21, 1988.

[3] A. Jabbour and S. Sabourau. Sharp upper bounds on the length of the shortest closed geodesic on complete punctured spheres of finite area. 2020. Preprint.

[4] Y. Liokumovich, A. Nabutovsky, and R. Rotman. Lengths of three simple periodic geodesics on a Riemannian 2-sphere. *Mathematische Annalen*, 367:831–855, 2017.

[5] A. Nabutovsky and R. Rotman. The length of the shortest closed geodesic on a 2-dimensional sphere. *International Mathematics Research Notices*, 2002(23):1211–1222, 2002.

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[7] S. Sabourau. Filling radius and short closed geodesics of the 2-sphere. *Bulletin de la Société Mathématique de France*, 132(1):105–136, 2004.

[8] S. Sabourau. Local extremality of the Calabi–Croke sphere for the length of the shortest closed geodesic. *Journal of the London Mathematical Society*, 82(3):549–562, 2010.

[9] G. Thorbergsson. Closed geodesics on non-compact Riemannian manifolds. *Mathematische Zeitschrift*, 159(3): 249–258, Oct 1978.